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Conformal Invariance and the Six-Dimensional Formalism†

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Abstract

Conformal invariance is discussed assuming the equations are well defined in arbitrary coordinate systems. This assumption leads to some constraints on scale dimensions of terms, and constraints on the introduction of 'conformally invariant massive equations'. The six-dimensional formalism is then discussed, and is generalized to project to all conformally flat spaces. Finally the imbedding of Minkowski space equations is studied. $SO(4, 2)$ breaking is seen to enter due to the presence of a non-invariant scalar field, and a non-invariant vector field. The theorem relating invariance of the six-space equations under $SO(4, 2)$ to the invariance of their corresponding four-space equations under the conformal group is carefully stated and proved.

1. *Introduction*

The conformal group and conformal invariance are defined for a general space-time, assuming the field equations and fields are tensor densities under general change of coordinates. This assumption puts restrictions on the transformation properties under the Minkowski space conformal group of terms in the equations. In particular it implies that all terms in the equation must have the same scale dimension. It also makes questionable the introduction of 'conformally invariant masses'.

Conformal invariance is then examined using the six-dimensional space in which the conformal group is the rotation group $SO(4, 2)$. (This was first done in Dirac's (1936) classic paper.) The standard projection to Minkowski space is generalized to a projection into any conformally flat space, using the standard methods of projecting into hyper-surfaces and quotient spaces.

The transformation of these projections under coordinate transformations is carefully discussed, and conditions are formulated in a coordinate independent

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way. It is found that unlike many similar cases, the projection of the metric generated covariant derivative is not the covariant derivative generated by the projection of the metric. An affinity that projects to the proper covariant derivative is given.

Using the formalism developed the imbedding of Minkowski space equations in six dimensions is studied. It is shown that in terms of six-space structures there are only two distinct ways to break conformal invariance, corresponding to the addition of mass-like terms, and to derivative couplings. From the imbedding of the Lie derivative of a field, the theorem relating to invariance of the imbedded field equations under $SO(4, 2)$ and the Minkowski space equations under the conformal group is proven. It is found that the theorem must be stated more carefully than is usually done.

2. *Four-Space*

The infinitesimal generators of the conformal group or the group of conformal motions on an arbitrary Riemannian space with metric g_{ab} are defined as the set of vectors $\xi^a_{(i)}$, such that

$$\underset{\xi_{(i)}}{\mathscr{L}} g_{ab} = \nabla_b \xi_a + \nabla_a \xi_b = \sigma g_{ab} \tag{2.1}$$

where $\mathscr{L}_\xi$ is the Lie derivative operator, σ is an arbitrary scalar field and ∇_a is the covariant derivative (Yano, 1955). The case where $\sigma \equiv 0$ gives the group of motions of the space. For Minkowski space M_4 this is the Poincaré group. In an arbitrary n-dimensional space the group of conformal motions may range from order 0 to order $\frac{1}{2}(n + 1)(n + 2)$. The maximum order is found in conformally flat spaces.

Conformally flat spaces are those whose metric differs from a flat space metric by a multiplicative scalar field. That is for the case of ordinary space-times $g_{ab} = \rho^2 \eta_{ab}$, where ρ is an arbitrary scalar field, and η_{ab} is the Minkowski space metric. All spaces whose metrics differ only by a multiplicative scalar field have the same group of conformal motions. Changing a metric by a multiplicative scalar field is called a conformal transformation of the metric. It is easily seen from the form of the Lie derivative that $\sigma = \frac{1}{2}\nabla_a \xi^a$.

In spaces conformal to Minkowski space (label them C_4's) the conformal group is given in terms of its four well known subgroups:

the six parameter, m_{ab}, homogeneous Lorentz group,

$$y'^a = e^{m^a{}_b} y^b$$

with generators

$$\xi^{(a)}_{(i,j)} = y^b (\delta^a_{(i)} \delta^{(j)}_b - \delta^a_{(j)} \delta^{(i)}_b) \tag{2.2}$$

the four parameter, t^a, group of translations,

$$y'^a = y^a + t^a$$
$$\xi^a_{(i)} = \delta^a_{(i)} \tag{2.3}$$

the four parameter, u^a, group of uniform accelerations,

$$\begin{aligned} y'^a &= \frac{[y^a + (\eta_{mn} y^m y^n) u^a]}{[1 + 2\eta_{rs} y^r u^s + (\eta_{rs} u^r u^s)(\eta_{fg} y^f y^g)]} \\ \xi^a_{(i)} &= (\eta_{rs} y^r y^s)\,\delta^a_{(i)} - 2y^a \eta_{r(i)} y^r \end{aligned} \tag{2.4}$$

and the one parameter, k, group of dilations,

$$\begin{aligned} y'^a &= e^k y^a \\ \xi^a &= y^a \end{aligned} \tag{2.5}$$

We will take a theory as invariant under the conformal group if solutions of the field equations of the theory are mapped into solutions of the field equations by the conformal group. For a theory in which the solutions are linear geometrical objects $\Phi^{\Lambda}_{(i)}$ (Yano, 1955), satisfying the field equations

$$O^{\pi}_{(j)}[\Phi^{\Lambda}_{(i)}] = 0 \tag{2.6}$$

there is a natural mapping of $\Phi^{\Lambda}_{(i)}$ induced by the group and this requirement takes the form (Anderson, 1967)

$$\begin{aligned} &O^{\pi}_{(j)}[\Phi^{\Lambda}_{(i)}] = 0 \qquad \text{implies} \\ &O^{\pi}_{(j)}[\Phi^{\Lambda}_{(i)} + \epsilon \underset{\xi}{\mathscr{L}}\, \Phi^{\Lambda}_{(i)}] = 0 \end{aligned} \tag{2.7}$$

where ϵ is an infinitesimal parameter and ξ^a an arbitrary generator of the conformal group. Scalars, vectors, tensors and tensor densities are linear geometrical objects. On the other hand spinors, as they are defined on the frame bundle over the space not the space itself, are not by the standard definitions geometrical objects and thus have no straightforward natural mapping under a Lie group defined on the space. A discussion of their invariance is therefore mathematically more ambiguous and will not be covered here. It would also be possible to define other mappings of the $\Phi^{\Lambda}_{(i)}$ under the conformal group and use these rather than the natural mapping to define an invariance, but we will also not consider these possibilities.

For illustrative purposes we will look at one curved space generalization of the zero rest mass Klein-Gordon equation. We will restrict ourselves to the standard four-dimensional space-time of general relativity. This equation reads (Penrose, 1964)

$$\left(g^{ab}\nabla_a\nabla_b + \frac{R}{6}\right)S = 0 \tag{2.8}$$

where S is taken to be a scalar density of weight W, and R is the curvature scalar. [The sign conventions used are throughout this paper those of Anderson (Anderson, 1967).] Then we wish to examine if

$$\left(g^{ab}\nabla_a\nabla_b + \frac{R}{6}\right)(S + \epsilon \underset{\xi}{\mathscr{L}}\, S) = 0 \tag{2.9}$$

For this we will need for the conformal group the commutator of the Lie and the covariant derivatives acting on a tensor density of weight W. This is given by (Yano, 1955)

$$[\underset{\xi}{\mathscr{L}}, \nabla_a] T^{b\cdots}_{c\ldots} = \Sigma^b_{af}\, T^{f\cdots}_{c\ldots} - \Sigma^f_{ca}\, T^{b\cdots}_{f\ldots} + \cdots - \cdots - W\, \Sigma^f_{fa}\, T^{b\cdots}_{c\ldots} \quad (2.10)$$

where

$$\sum_{af}^{b} = \tfrac{1}{2}(\delta_a{}^b \sigma_f + \delta_f{}^b \sigma_a - \sigma^b g_{af}) \quad (2.11)$$

$$\sigma_a \equiv \nabla_a \sigma \quad (2.12)$$

We see that the commutator vanishes for the special case of motions, and also when T is a scalar field, but does not in general vanish even in flat space for conformal motions which have non-constant σ. We will also need that (Yano, 1955)

$$\nabla_a \sigma^a = \tfrac{1}{3}[\underset{\xi}{\mathscr{L}} R + \sigma R] \quad (2.13)$$

We can now compute

$$\underset{\xi}{\mathscr{L}}\left[\left(g^{ab}\nabla_a\nabla_b + \frac{R}{6}\right)S\right] = 0$$

$$= -\sigma g^{ab}\nabla_a\nabla_b S + g^{ab}\, \underset{\xi}{\mathscr{L}}\, (\nabla_a\nabla_b S)$$

$$+ (\underset{\xi}{\mathscr{L}}\, R)\, \frac{S}{6} + \frac{R}{6}\, \underset{\xi}{\mathscr{L}}\, S$$

which using equation (2.10), (2.11) and (2.13) gives

$$-\sigma g^{ab}\nabla_a\nabla_b S + g^{ab}\nabla_a\nabla_b(\underset{\xi}{\mathscr{L}} S) + \frac{R}{6}\underset{\xi}{\mathscr{L}} S + (\underset{\xi}{\mathscr{L}} R)\frac{S}{6}\, 4W\sigma^a\nabla_a S$$

$$+ \sigma^a\nabla_a S - \tfrac{2}{3} WS(\underset{\xi}{\mathscr{L}} R + \sigma R) = 0$$

so that for $W = \frac{1}{4}$

$$\left(g^{ab}\nabla_a\nabla_b + \frac{R}{6}\right)(\underset{\xi}{\mathscr{L}} S) = 0 \quad (2.14)$$

This result also holds in the special case of flat space where $R = 0$ and in a rectilinear coordinate system the covariant derivative is the partial derivative. Thus we have that for the ordinary zero-mass Klein-Gordon equation it is conformally invariant if and only is S is considered to be a scalar density of weight $\frac{1}{4}$.

If invariance is required just under homothetic motions, that is conformal motions with σ a constant scalar field, then we have no restriction on the weight of S. In Minkowski space the dilation subgroup of the conformal group has $\sigma = 2K$ so that it is a homothetic motion.

Continuing in M_4, we consider other terms added to equation (2.8) (Carruthers, 1970) so that we have the equation

$$\eta^{ab}\nabla_a\nabla_b S + m^2 S - 3gS^2 - 4fS^3 = 0 \tag{2.15}$$

We first note that for equation (2.15) to hold in an arbitrary coordinate system, and thus to have a geometric meaning and also to be extendable to situations where general relativity is necessary, all the terms must transform in the same way under mappings of space-time onto itself. That is if S is a scalar density of weight W, $\eta^{ab}\nabla_a\nabla_b S$ is a scalar density of weight W, so that m must be a scalar field (a scalar density of weight 0), g a scalar density of weight $-W$ and f a scalar density of weight $-2W$.

For S of weight $\frac{1}{4}$, $m = 0, g = 0$ equation (2.15) is invariant under the 15 parameter conformal group of Minkowski space. But if we want f as m is to be a constant in all coordinate systems we must take S to be of weight 0. In this case we are breaking conformal invariance in a very subtle way by changing the type of geometrical object in our field equation.

At various times the idea of a conformally invariant mass term has been raised (Barut & Haugen, 1972). That is allowing the mass to transform under the conformal group in such a way as to make equations with mass type terms invariant under the group. The requirement that all terms in a field equation transform in the same way limits and in some cases makes this idea impossible. In the case of equation (2.15) if the mass is a constant in a rectilinear coordinate system it must stay the same constant under all groups of transformations and in particular cannot pick up powers of σ under conformal transformations. That is, under our definitions the massive Klein-Gordon equation can never be made conformally invariant.

The scale dimension of a classical field is defined (Carruthers, 1971) as l if for the coordinate transformation

$$T'^{a\ldots}_{b\ldots} = \rho^l T^{a\ldots}_{b\ldots} \tag{2.16}$$

$$y'^a = \rho y^a \tag{2.17}$$

For a tensor density of weight W

$$T'^{a\ldots}_{b\ldots} = J^W \underbrace{\overbrace{T^{m\ldots}_{n\ldots}}^{i}}_{j} \frac{\partial y'^a}{\partial y^m} \frac{\partial y^n}{\partial y'^b} \tag{2.18}$$

Using equation (2.17)

$$J = \rho^{-4}, \qquad \frac{\partial y'^a}{\partial y^m} = \rho\delta_m{}^a \tag{2.19}$$

so that

$$T'^{a\ldots}_{b\ldots} = \rho^{-(4W+i-j)} T^{a\ldots}_{b\ldots} \tag{2.20}$$

Thus S has a scale dimension $-4W = -1$ for equation (2.8) to be conformally invariant.

It is a necessary condition for equations to have a geometric meaning, that every term has the same scale dimension. This requirement seems to be commonly violated (Carruthers, 1971). Carruthers also does not seem to consider the transformation of g_{ab} in computing scale dimension which is inconsistent with our definition, so there is perhaps some confusion over what is the definition of scale dimension.

If we examine Maxwell's source free equations

$$g^{ab} \nabla_a F_{bd} = 0 \tag{2.21}$$

$$\nabla_{[a} F_{bd]} = 0 \tag{2.22}$$

we also find that if F_{bd} is a solution to Maxwell's equations then $\mathscr{L}_\xi F_{bd}$ is a solution, if F_{bd} is a tensor field, that has weight 0.

3. *Six-Space Formalism*

The study of the conformal group of Minkowski space and conformal invariance is illuminated by utilising the well known relationship between this group and the rotation group $SO(4, 2)$ in the six-dimensional flat space R_6 with a metric g_{AB} of signature -2. That is there exists a coordinate system in which $g_{AB} = \mathrm{diag}(+1, -1, -1, -1, -1, +1)$. (Grgin, 1968; see also for further references and discussion Mack & Salam, 1969; and Barut & Haugen, 1972). We will follow Grgin in exhibiting the relationship between the groups, but then obtain the relationship between geometrical structures and field equations in R_6 and an arbitrary C_4, rather than just R_6 and M_4. We will through this approach obtain a clearer insight into the geometrical nature of this relationship.

The relationship between R_6 and C_4 is most easily exhibited by using a partially null coordinate system in which the metric has the form

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \begin{array}{|cc} 0 & 2 \\ 2 & 0 \end{array} \end{pmatrix}, \qquad \eta^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \begin{array}{|cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \end{pmatrix} \tag{3.1}$$

Upper case Latin index letters run from 0 to 5, and lower case letters from 0 to 3. η_{ab} is the Minkowski metric, $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. We label the R_6 coordinates by $x^A = (x^a, x^4, x^5)$, and we will let $x^4 = \Omega$ and $x^5 = \Psi$.

The tangent vectors to the coordinate lines x^A are given by

$$e^A_{(b)} = \frac{\partial x^A}{\partial x^b} = \delta_b{}^A, \qquad e^A_{(\Omega)} = \frac{\partial x^A}{\partial \Omega} = \delta_4{}^A, \qquad e^A_{(\Psi)} = \frac{\partial x^A}{\partial \Psi} = \delta_5{}^A \quad (3.2)$$

$$e_{A(B)} \equiv \eta_{AD} e^D_{(B)} \quad (3.3)$$

$$\eta_{AB} e^A_{(E)} e^B_{(F)} = \eta_{(EF)} \quad (3.4)$$

so that

$$\begin{aligned} e_{A(\Omega)} e^A_{(\Omega)} &= 0 = e_{A(\Psi)} e^A_{(\Psi)} \\ e_{A(\Psi)} e^A_{(\Omega)} &= 2 \end{aligned} \quad (3.5)$$

We also define

$$y^a \equiv \frac{x^a}{\Omega} \quad (3.6)$$

and

$$\Phi \equiv \eta_{AB} x^A x^B = \eta_{ab} x^a x^b + 4\Omega\Psi \quad (3.7)$$

The surface $\Phi = 0$ is a null cone with vertex at $x^A = 0$, and will be labeled N_5.

A general element of $SO(4, 2)$ is given by

$$\begin{aligned} x'^A &= E^A{}_B x^B \\ E^A{}_B &= \exp(\eta^{AD} \epsilon_{DB}) \end{aligned} \quad (3.8)$$

where ϵ_{DB} is an arbitrary constant skew-symmetric matrix. The transformation $x'^A = E^A{}_B x^B$ induces a non-linear transformation on the y^a's given by

$$y'^a = \frac{x'^a}{\Omega'} = \frac{E_B{}^a x^B}{E_F{}^4 x^F} \quad (3.9)$$

When we restrict the x^A's to the surface $\Phi = 0$, which is mapped into itself by a rotation, the induced transformation of the y^a's is identical to the action of the C_4 conformal group in a rectilinear coordinate system.

The infinitesimal generators ϵ_{BD} can be expanded in terms of the basis set $e^A_{(F)}$, and this is the decomposition that leads to a direct identification with the various subgroups of the conformal group on C_4. For

$$\epsilon_{BD} = M^{bd} e_{(b)B} e_{(d)D} \quad (3.10)$$

where M^{bd} is an arbitrary anti-symmetric tensor

$$x'^A = x^A + \eta^{AB} M^{bd} e_{(b)B} e_{(d)D} x^D$$

$$y'^a = \frac{x'^a}{\Omega'} = \frac{x^a + M^a{}_d x^d}{\Omega} = y^a + M^a{}_d y^d \quad (3.11)$$

where we raise and lower lower case Latin indices with the Minkowski metric. Equation (3.11) is in the form of an infinitesimal homogeneous Lorentz transformation, so that ϵ_{BD} in equation (3.10) corresponds to the Lorentz subgroup of the conformal group.

Choosing:

$$\epsilon_{DB} = t^a e_{(a)[D}e_{B](\Psi)} \tag{3.12}$$

implies $y'^a = y^a + t^a$,

$$\epsilon_{DB} = \theta e_{(\Omega)[D}e_{B](\Psi)} \tag{3.13}$$

implies

$$y'^a = \frac{1}{(1+\theta)} y^a$$

and

$$\epsilon_{DB} = 4u^a e_{(\Omega)[D}e_{B](a)} \tag{3.14}$$

implies

$$y'^a = y^a + u^b[\eta_{ef}y^e y^f \delta_b{}^a - 2y^a \eta_{bf} y^f]$$

It is interesting to note that it is only in the case of the uniform accelerations equation (3.14) that we need to make use of the restriction $\Phi = 0$.

We will now proceed to relate geometrical objects on R_6 to corresponding objects on C_4. To make this relation we will use the fact that we have a relation in the y^a's between our R_6 coordinate system and a rectilinear C_4 coordinate system. To make this relationship we formally introduce a new coordinate system on R_6

$$y^A = (y^a, y^4, y^5) \tag{3.15}$$

where we need to introduce some criteria to specify

$$y^4 \equiv \omega = f(x^A) \tag{3.16}$$

$$y^5 \equiv h(x^A) \tag{3.17}$$

From the group representation mappings, we have that the relationship between the spaces should be the product of projecting into N_5 and then the projection into the quotient space whose points are the curves y^a = const. (For a general discussion of hypersurface projection see Eisenhart, 1927. Particular cases of projections into quotient spaces are found in Bergmann, 1942 and Geroch, 1971 and 1972.) It is standard in projecting into hypersurfaces to take one of our new coordinates to be given by the equation of the hypersurface. Thus we take

$$y^5 = h(x^A) = \Phi \tag{3.18}$$

Of course any function $f(\Phi)$ that vanished if and only if Φ vanished would do as well, but doing this would make no significant change in our relationships. The (y^a, y^4) are now a set of coordinates for points in N_5. The projection into a quotient space is basically an operation in classical projective geometry. It is standard in this field to take $f(x^A)$ to be a homogeneous function of non-zero degree. We will take $f(x^A)$ to be a function only of (x^a, Ω) and to be homogeneous of degree 1. Some other degree of homogeneity would again not make a significant difference. $f(x^A)$ is taken independent of Ψ as Ψ is the coordinate we eliminate by the $\Phi = 0$ condition, x^a and Ω being needed to construct the quotient space.

We can now proceed to invert the equation set (3.6), (3.16) and (3.18). We first note

$$y^4 \equiv \omega = f(x^a, \Omega) = \Omega f\left(\frac{x^a}{\Omega}, 1\right) \equiv \Omega \bar{f}(y^a)$$

or

$$\Omega = \frac{\omega}{\bar{f}} \tag{3.19}$$

implies

$$x^a = \frac{\omega}{\bar{f}} y^a \tag{3.20}$$

$$\Psi = \frac{\bar{f}\Phi}{4\omega} - \frac{\omega y^a y_a}{4\bar{f}} \tag{3.21}$$

It is also useful at this time to define

$$K_a{}^A \equiv \frac{\partial x^A}{\partial y^a} \tag{3.22}$$

$$\nu^A \equiv \frac{\partial x^A}{\partial \omega} \tag{3.23}$$

$$\tau^A \equiv \frac{\partial x^A}{\partial \Phi} \tag{3.24}$$

$$K_A{}^a \equiv \frac{\partial y^a}{\partial x^A} \tag{3.25}$$

$$f_A \equiv \frac{\partial f}{\partial x^A} \tag{3.26}$$

$$\Phi_A \equiv \frac{\partial \Phi}{\partial x^A} \tag{3.27}$$

$$J_0 \equiv \det\left[\frac{\partial x^A}{\partial y^B}\right] \equiv \left|\frac{\partial x^A}{\partial y^B}\right| \tag{3.28}$$

We note that $K_a{}^A$, ν^A, τ^A, are contravariant vectors and $K_A{}^a$, f_A, Φ_A covariant vectors under change in the R_6 coordinates, while $K_a{}^A$ is a covariant and $K_A{}^a$

a contravariant vector under change in the C_4 coordinate system. $(K_a{}^A, \nu^A, \tau^A)$ forms a set of basis vectors on R_6, whose dual basis set is $(K_A{}^a, f_A, \Phi_A)$. That is

$$K_A{}^a K_b{}^A = \delta_b{}^a, \qquad \nu^A f_A = 1, \qquad \Phi_A \tau^A = 1 \tag{3.29}$$

and all other products are zero. Also

$$\Phi^A \Phi_A = 4\Phi, \qquad \tau^A \tau_A = 0, \qquad \nu^A \nu_A = -\Phi/f^2 \tag{3.30}$$

A geometrical object defined on C_4 will just be a function of the y^a's. That is it must correspond to an object in R_6 which on N_5 is a homogeneous function of (x^a, Ω). We will further assume that the objects are homogeneous in x^A everywhere. In the neighborhood of N_5 we can expand any homogeneous scalar field which can correspond to a scalar field on C_4 in a power series in Φ given by

$$S = S_0(x^a, \Omega) + \Phi S_1(x^a, \Omega) + \cdots + \Phi^n S_n(x^a, \Omega) + \ldots \tag{3.31}$$

where if S is homogeneous of degree h, S_n is homogeneous of degree $h - 2n$

$$S_0(x^a, \Omega) = \Omega^n S_0(y^a, 1) = \Omega^n \bar{S}_0(y^a) = \omega^n \left[\frac{1}{\bar{f}^n} \bar{S}_0(y^a) \right] \tag{3.32}$$

For $n \neq 0$ we have two natural choices for a scalar field on $C_4 S_4(y^a)$ given by $\bar{S}_0$ and $\bar{S}_0/\bar{f}^n = \hat{S}_0$. In the special case of $M_4, \bar{f} = 1$ and $\hat{S}_0$ and $\bar{S}_0$. We will see that if we wish to treat the metric in the same way as all other tensors we should choose $\hat{S}_0$, but we should keep in mind that in some cases it will be necessary or may be desirable to use $\bar{f}^m \bar{S}^0$ where m has some other value. This ambiguity is not removable as the geometrical character of an object does not necessarily completely determine how it should transform when the metric of the space is conformally transformed.

The inverse mapping of a scalar field on C_4, $S_{(4)}(y^a)$, is some element S of the equivalence class of scalar fields on R_6 that have

$$\Omega^n S_{(4)} \left(\frac{x^a}{\Omega} \right) \bar{f}^l \left(\frac{x^a}{\Omega} \right) = S_0(x^a, \Omega)$$

where n and l are arbitrary. n is chosen so as to determine the degree of homogeneity of S.

For any tensor density $T^{A\cdots}_{B\cdots}$ of weight W on R_6 we can define

$$T^{a\cdots}_{b\cdots} = J_0{}^W K_A{}^a K_b{}^B \cdots T^{A\cdots}_{B\cdots} \tag{3.33}$$

which is a set of scalars of weight 0 on R_6. In our x^A coordinate system

$$\tau^A = \frac{\delta_5{}^A}{4\Omega} \tag{3.34}$$

$$\Phi^A = 2x^A \tag{3.35}$$

$$\nu^A = \frac{\Phi^A}{2f} - \frac{2\Phi}{f} \tau^A \tag{3.36}$$

$$K_A{}^a = \frac{1}{\Omega}(\delta_A{}^a - \Phi^a \tau_A) \tag{3.37}$$

$$K_a{}^A = \Omega(\delta_a{}^A - f_a \nu^A - \tau^A \Phi_a) \tag{3.38}$$

$$J_0 = \frac{\Omega^4}{4f} \tag{3.39}$$

so that if $T_{B\ldots}^{A\cdots}$ has k contravariant indices and l covariant indices and is homogeneous of degree n, $T_{b\ldots}^{a\cdots}$ is homogeneous of degree $(n + 3W + l - k) = r$. Each homogeneous $T_{b\ldots}^{a\cdots}$ as it is a scalar can be mapped into a function $T_{(4)b\ldots}^{a\cdots} = \bar{f}^m T_{0b\ldots}^{a\cdots}$ where m can be of any value. Under change in the C_4 coordinate system $T_{(4)b\ldots}^{a\cdots}$ transforms as a tensor density of weight W.

Given a tensor density of weight W on C_4, the inverse mapping is given by

$$T_{B\ldots}^{A\cdots} = J_0^{-W} K_a{}^A K_B{}^b \ldots \left(\Omega^n \bar{f}^l T_{(4)b\ldots}^{a\cdots}\left(\frac{x^a}{\Omega}\right) + \Phi T_{1\,b\ldots}^{\ a\cdots} + \ldots\right) + T_{\perp B\ldots}^{\ A\cdots} \tag{3.40}$$

where $T_{1\,b\ldots}^{\ a\cdots}$ is an arbitrary function of (x^a, Ω) and $T_{\perp B\ldots}^{\ A\cdots}$ is a tensor density of weight W, such that the contraction of at least one index with a projection operator $K_A{}^a$ or $K_b{}^B$ gives zero. That is

$$0 = K_A{}^a T_{\perp B\ldots}^{\ A\cdots} \qquad \text{or} \qquad 0 = K_b{}^B T_{\perp B\ldots}^{\ A\cdots} \ \text{or} \ldots \tag{3.41}$$

As

$$(\Omega^n \bar{f}^l T_{(4)b\ldots}^{a\cdots} + \Phi T_{1\,b\ldots}^{\ a\cdots} + \ldots) = T_{b\ldots}^{a\cdots} \tag{3.42}$$

we have from equation (3.33) that

$$T_{B\ldots}^{A\cdots} = K_E{}^A K_B{}^F \ldots T_{F\ldots}^{E\cdots} + T_{\perp B\ldots}^{\ A\cdots} \tag{3.43}$$

$$K_E{}^A = K_a{}^A K_E{}^a = \delta_E{}^A - f_E \nu^A - \tau^A \Phi_E \tag{3.44}$$

If we define a homogeneous function $\overset{K}{T}{}_{B\ldots}^{A\cdots}$ by

$$\overset{K}{T}{}_{B\ldots}^{A\cdots} = K_E{}^A K_B{}^F \ldots T_{F\ldots}^{E\cdots} \tag{3.45}$$

then $\overset{K}{T}{}_{B\ldots}^{A\cdots}$ represents a C_4 tensor in R . $K_E{}^A$ is a projection operator so that

$$K_E{}^A K_B{}^E = K_B{}^A \tag{3.46}$$

and thus acts as $\delta_B{}^A$ on $\overset{K}{T}{}_{B\ldots}^{A\cdots}$ type tensors.

The projection operator from R_6 to C_4 can be written as the product of a projection operator into N_5 and a projection operator into the quotient space of curves y^a = const. If we define the hypersurface projection operators

$$h_\alpha{}^A = \delta_\alpha{}^A - \tau^A \Phi_\alpha \tag{3.47}$$

$$h_A{}^\alpha = \delta_A{}^\alpha \tag{3.48}$$

and the quotient space projection operators

$$J_\alpha{}^a = \frac{1}{\Omega}\left[\delta_\alpha{}^a - x^a \frac{\delta_\alpha{}^4}{\Omega}\right] \tag{3.49}$$

$$J_a{}^\alpha = \Omega\left[\delta_a{}^\alpha - \frac{1}{f} x^\alpha \frac{\partial f}{\partial x^a}\right] \tag{3.50}$$

where α runs from 0 to 4, then

$$K_a{}^A = J_a{}^\alpha h_\alpha{}^A \tag{3.51}$$

$$K_A{}^a = J_\alpha{}^a h_A{}^\alpha \tag{3.52}$$

We obtain the interpretation of $f(x^a, \Omega)$ in our formalism by projecting the metric η_{AB} of R_6 into C_4

$$g_{ab} \equiv K_a{}^A K_b{}^B \eta_{AB} = \Omega^2 \eta_{ab} = \frac{\omega^2}{\bar{f}^2}\eta_{ab} \tag{3.53}$$

so that

$$\bar{g}_{ab} = \frac{1}{\bar{f}^2}\eta_{ab} \tag{3.54}$$

gives an arbitrary C_4 metric with conformal factor $\bar{f}^{-2}$. One can choose to give various 'physical' interpretations to the ω that appears in g_{ab}. See for example Barut and Haugen (1972) or Kastrup (1966). From (3.54) we see that taking

$$\omega = f(x^a, \Omega) = \Omega \tag{3.55}$$

in our projection operators restricts them to projecting into flat space. With this restriction to flat space it is easily seen that our prescription for relating six-dimensional structures to four-dimensional structures reduces to essentiallly the same form as is found in the previously cited works.

We define ξ^A to be an arbitrary generator of $SO(4, 2)$ so that

$$\xi^A = \eta^{AB}\epsilon_{BE}x^E \tag{3.56}$$

Then

$$\xi^A = \xi_0{}^A + \Phi\xi^A_{,S}\tau^S \tag{3.57}$$

$$\xi_0{}^A = \eta^{AB}\epsilon_{BE}\left[\delta_e{}^E x^e + \delta_4{}^E\Omega - \frac{\delta_5{}^E}{4\Omega}(\eta_{rs}x^r x^s)\right] \tag{3.58}$$

It is then easy to check that

$$\xi^a = K_A{}^a \xi_0{}^A \tag{3.59}$$

is a homogeneous function of degree zero in x^A and is an arbitrary generator of the conformal group, as it must be if the projection from R_6 to C_4 is consistent.

We have implicitly used three types of coordinate transformations in relating R_6 to C_4. We will now look at these in more detail. The first type of coordinate transformation is a change in the R_6 coordinate system but no change in the C_4 system or in the relationship between R_6 and C_4. That is

$$x'^A = \mathscr{F}^A(x), \qquad x^A = \underline{\mathscr{F}}^A(x') \tag{3.60}$$

Then if

$$\begin{aligned} \tilde{y}^a &= \frac{\underline{\underline{\mathscr{F}}}^a}{\underline{\mathscr{F}}^4} = y^a \\ \tilde{y}^4 &= y^4 = f'(x') = f(x) \\ \tilde{y}^5 &= y^5 = \Phi'(x') = \Phi(x) \end{aligned} \tag{3.61}$$

all our projection operators transform as vectors and $T_{(4)b\cdots}^{a\cdots}$ transforms as a scalar.

The second type of transformation is a coordinate transformation on C_4, but no change in R_6 or the relationship between them. That is

$$\begin{aligned} \tilde{y}^a &= \mathscr{G}^a(y^b), \qquad y^a = \underline{\mathscr{G}}^a(\tilde{y}) = \frac{x^a}{\Omega} \\ \tilde{y}^4 &= y^4 = f(x), \qquad \tilde{y}^5 = y^5 = \Phi \end{aligned} \tag{3.62}$$

Then $T_{(4)b\cdots}^{a\cdots}$ transforms as the appropriate type of tensor density. That is the method we have for relating R_6 with C_4 is independent of what coordinate systems we work in, even though we have evaluated the relations in a simple set of coordinates.

The third type of coordinate transformation is an induced coordinate transformation. That is

$$\begin{aligned} x'^A &= \mathscr{F}^A(x) \\ \tilde{y}^a &= \frac{\mathscr{F}^a}{\mathscr{F}^4} \\ \tilde{y}^4 &= \tilde{f}(x') = \Omega'\tilde{f}\left(\frac{x'^a}{\Omega'}, 1\right) \equiv \Omega'\tilde{\tilde{f}}(\tilde{y}) \\ \tilde{y}^5 &= \Phi \end{aligned} \tag{3.63}$$

If $\mathscr{F}^a/\mathscr{F}^4|_{\Phi=0}$ is just a function of the y^a's this generates a coordinate transformation on C_4. That is given a tensor density $T_{(4)b\cdots}^{a\cdots}$ on C_4

$$\tilde{T}_{(4)b\cdots}^{a\cdots} = \left|\frac{\partial y^r}{\partial \tilde{y}^t}\right|^W \frac{\partial \tilde{y}^a}{\partial y^e}\frac{\partial y^f}{\partial \tilde{y}^b}\cdots T_{(4)f\cdots}^{e\cdots} \tag{3.64}$$

We can now also consider the quotient space in N_5 defined by the curves $\tilde{y}^a$ = constant, and the projection of tensors into it. We thus define

$$\bar{T}'^{a\cdots}_{b\cdots} \equiv \tilde{\tilde{f}}^m \left|\frac{\partial x'^R}{\partial \tilde{y}^T}\right|^W \frac{\partial \tilde{y}^a}{\partial x'^A}\frac{\partial x'^B}{\partial \tilde{y}^b} T'^{A\cdots}_{B\cdots}\Bigg|_{\substack{\Phi=0\\ \tilde{\omega}=1}} \tag{3.65}$$

$\bar{T}'^{a\cdots}_{b\cdots}$ is a function of just the $\tilde{y}^a$'s and can be taken as a tensor density of weight W under change in $\tilde{y}^a$ coordinates. We can thus examine when starting with the tensor density $T^{A\cdots}_{B\cdots}(x^E)$ under what conditions

$$\tilde{T}_{(4)b\cdots}^{a\cdots} = \bar{T}'^{a\cdots}_{b\cdots} \tag{3.66}$$

We have using the relationship between x' and x

$$\bar{T}'^{a\cdots}_{b\cdots} = \tilde{\bar{f}}^m \left|\frac{\partial y^R}{\partial \tilde{y}^T}\right|^W \left|\frac{\partial x^E}{\partial y^F}\right|^W \frac{\partial \tilde{y}^a}{\partial x^A}\frac{\partial x^B}{\partial \tilde{y}^b}\cdots T^{A\cdots}_{B\cdots}\Bigg|_{\substack{\Phi=0\\ \tilde{\omega}=1}} \tag{3.67}$$

and

$$\tilde{T}_{(4)b\cdots}^{a\cdots} = \left|\frac{\partial y^e}{\partial \tilde{y}^f}\right|^W \frac{\partial \tilde{y}^a}{\partial x^A}\frac{\partial x^A}{\partial y^e}\tilde{\tilde{f}}^m \left|\frac{\partial x^R}{\partial y^T}\right|^W \frac{\partial y^e}{\partial x^E}\cdots T^{E\cdots}_{F\cdots}\Bigg|_{\substack{\Phi=0\\ \tilde{\omega}=1}} \tag{3.68}$$

where

$$\tilde{\tilde{f}}(\tilde{y}^a) = \bar{f}(y^a)$$

Thus we get immediately from the Jacobians and the dependence on $\tilde{\tilde{f}}$ and $\tilde{\bar{f}}$ that

$$\tilde{y}^4 = \tilde{f}(x'^A) = f(x^A) = y^4 \tag{3.69}$$

for (3.66) to hold. We also need

$$\frac{\partial \tilde{y}^a}{\partial x^A} T^A = \frac{\partial \tilde{y}^a}{\partial x^A} K_E{}^A T^E \Bigg|_{\Phi=0} \tag{3.70}$$

and

$$\frac{\partial x^A}{\partial \tilde{y}^a} T_A = \frac{\partial x^A}{\partial \tilde{y}^a} K_A{}^E T_E \Bigg|_{\Phi=0} \tag{3.71}$$

Thus it is sufficient that $\partial x^A/\partial \tilde{y}^a$ and $\partial \tilde{y}^a/\partial x^A$ are members of the set of $\overset{K}{T}{}^{\cdots}_{\cdots}$ type tensors or that $T^{A\cdots}_{B\cdots}$ is a member of the set of $\overset{K}{T}{}^{\cdots}_{\cdots}$ type tensors when evaluated on N_5.

It can be verified that when $\mathscr{F}^A(x)$ is an arbitrary element of $SO(4, 2)$ $\partial x^A/\partial \tilde{y}^a|_{\Phi=0}$ is not a $\overset{K}{T}{}^{\cdots}_{\cdots}$ type tensor, due to the elements of the group that are identified with the uniform accelerations. It is also true that for the rotations considered as an active group mapping the space of tensor densities onto itself the set of $\overset{K}{T}{}^{\cdots}_{\cdots}$ type tensors is not mapped into itself, so we must look closer at the induced transformations of the rotation group.

In infinitesimal form an element of $SO(4, 2)$ is given by

$$\begin{aligned} x'^A &= x^A + \eta^{AB}\epsilon_{BT}x^T = x^A + \xi^A \\ x^A &= x'^A - \eta^{AB}\epsilon_{BT}x'^T \end{aligned} \tag{3.72}$$

Putting equation set (3.72) into (3.70) and (3.71) results in the sufficient condition for the satisfaction of equation (3.66) is that

$$T^{A\cdots}_{B\cdots}\Phi_A = T^{A\cdots}_{B\cdots}\Phi^B = \ldots = 0 \tag{3.73}$$

The set of tensors that satisfy (3.73) goes into itself under the mapping of the rotation group. The metric η_{AB} does not satisfy any of the previous conditions but it can be checked that for $SO(4, 2)$ it satisfies (3.66), when (3.69) holds.

As has been stated before some sort of homogeneity condition in x^A is needed for objects on R_6 associated with C_4 objects. Homogeneity conditions are related to intrinsically coordinate independent conditions involving Lie derivatives. If we have a tensor density $T^{A\cdots}_{B\cdots}$ on R_6 which is homogeneous of degree n in x^A, then

$$T^{A\cdots}_{B\cdots,S}x^S = nT^{A\cdots}_{B\cdots} \tag{3.74}$$

In the x^A coordinate system $\Phi^A = 2x^A$ so that

$$\tfrac{1}{2}\underset{\Phi^E}{\mathscr{L}}\, T^{A\cdots}_{B\cdots} = (T^{A\cdots}_{B\cdots,R}x^R - kT^{A\cdots}_{B\cdots} + lT^{A\cdots}_{B\cdots} + 6WT^{A\cdots}_{B\cdots}) \tag{3.75}$$

and

$$\underset{\Phi^E}{\mathscr{L}}\, T^{A\cdots}_{B\cdots} = +2(n - k + l + 6W)T^{A\cdots}_{B\cdots} \tag{3.76}$$

is equivalent to homogeneity in x^A of degree n.

If $T_{(4)}{}^{a\cdots}_{b\cdots}$ is a tensor density on C_4, then $T_{(4)}{}^{a\cdots}_{b\cdots}$ considered as a set of scalar fields on R_6 is just a function of the y^a's and satisfies

$$\frac{\partial T_{(4)}{}^{a\cdots}_{b\cdots}}{\partial\omega} = \underset{\nu^A}{\mathscr{L}}\, T_{(4)}{}^{a\cdots}_{b\cdots} = 0 \tag{3.77}$$

The set of all tensor densities $T^{A\cdots}_{B\cdots}$ on R_6 such that

$$T^{a\cdots}_{b\cdots} = J_0{}^W K_A{}^a K_b{}^B T^{A\cdots}_{B\cdots} \tag{3.78}$$

and

$$T_0{}^{a\cdots}_{b\cdots} = \frac{\omega^n}{f^m} T_{(4)}{}^{a\cdots}_{b\cdots} \tag{3.79}$$

where n can take any value but m is fixed, correspond to $T_{(4)}{}^{a\cdots}_{b\cdots}$. It is easily checked that the Lie derivatives with respect to ν^A of J_0, a scalar density of weight -1, and of $K_A{}^a$ and $K_b{}^B$, covariant and contravariant vectors respectively vanish, so that equation (3.77) implies a condition on the Lie derivatives of $T^{A\cdots}_{B\cdots}$'s that are mapped into C_4. In the x^A coordinate system

$$\begin{aligned}\underset{\nu^E}{\mathscr{L}}\, T^{A\cdots}_{B\cdots}|_{\Phi=0} &= T^{A\cdots}_{B\cdots,R}x^R - kT^{A\cdots}_{B\cdots} + lT^{A\cdots}_{B\cdots} + 3WT^{A\cdots}_{B\cdots} \\ &= \underset{\Phi^E}{\mathscr{L}}\, T^{A\cdots}_{B\cdots} - 3WT^{A\cdots}_{B\cdots}\end{aligned} \tag{3.80}$$

The condition implied by equation (3.77) is

$$\underset{\nu E}{\mathscr{L}} T^{A\cdots}_{B\cdots}|_{\Phi=0} = \alpha T^{A\cdots}_{B\cdots} + J_{\perp}{}^{A\cdots}_{B\cdots} \tag{3.81}$$

where α is a constant. Thus the homogeneity condition (3.74) is stronger than necessary, but it seems to be the most convenient condition to impose. It also seems that the Lie derivative with respect to ν^A is the more relevant one to look at. From equation (3.80) it is clear that equation (3.74) also implies that on N_5 Lie derivatives with respect to ν^A take on a simple form.

To finish the relationship between R_6 geometry and C_4 geometry we need to find the relationship between the R_6 covariant derivative and the C_4 covariant derivative constructed using the C_4 metric $\hat{g}_{ab} = (1/\bar{f}^2)\eta_{ab}$. We will find this relationship for the case of tensor densities related by

$$T_{(4)}{}^{a\cdots}_{b\cdots} \underset{\substack{\Phi=0\\ \omega=1}}{=} \bar{f}^m T^{a\cdots}_{b\cdots} \underset{\substack{\Phi=0\\ \omega=1}}{=} \bar{f}^m J_0{}^W K_A{}^a K_b{}^B \ldots T^{A\cdots}_{B\cdots} \tag{3.82}$$

where the $T^{A\cdots}_{B\cdots}$'s are restricted to the set of $\overset{K}{T}{}^{A\cdots}_{B\cdots}$ type tensors.

We first calculate in the related x^A, y^a coordinate systems

$$\bar{f}^m J_0{}^W T^{A\cdots}_{B\cdots,T} K_t{}^T K_A{}^a K_b{}^B \ldots \equiv T^{a\cdots}_{b\cdots t} \tag{3.83}$$

We can use partial derivatives in (3.83) as in the x^A system the partial derivative and the covariant derivative are the same. Using our definitions of the projection operators, $\overset{K}{T}{}^{A\cdots}_{B\cdots}$ type tensors, and that noticing they imply

$$\omega_{,A} K_a{}^A = \Phi_{,A} K_a{}^A = 0 \tag{3.84}$$

$$K_D{}^a K^D_{d,B} K_b{}^B = -K^a_{D,B} K_d{}^D K_b{}^B \tag{3.85}$$

we get

$$T^{a\cdots}_{b\cdots t} \underset{\substack{\Phi=0\\ \omega=1}}{=} \frac{\partial T_{(4)}{}^{a\cdots}_{b\cdots}}{\partial y^t} - W J_{0,T} J_0^{-1} T_{(4)}{}^{a\cdots}_{b\cdots} - \frac{m}{\bar{f}} \frac{\partial \bar{f}}{\partial y^a} T_{(4)}{}^{a\cdots}_{b\cdots}$$
$$- T_{(4)}{}^{d\cdots}_{b\cdots} K_d{}^A K^a_{A,T} K_t{}^T \ldots + T_{(4)}{}^{a\cdots}_{d\cdots} K^d_{B,T} K_t{}^T K_b{}^B \ldots \tag{3.86}$$

From equations (3.34)-(3.39) we get

$$J_0^{-1} J_{0,A} K_a{}^A = -\frac{4}{\bar{f}} \frac{\partial \bar{f}}{\partial y^a} \tag{3.87}$$

and

$$K_d{}^A K^a_{A,T} K_t{}^T = \frac{\delta_d{}^a}{\bar{f}} \frac{\partial \bar{f}}{\partial y^t} + \frac{\delta_t{}^a}{\bar{f}} \frac{\partial f}{\partial y^d} \tag{3.88}$$

From our metric the Christoffel symbol of the second kind is given by

$$\Gamma^a_{bd} = -\frac{\delta_d{}^a}{\bar f}\frac{\partial \bar f}{\partial y^b} - \frac{\delta_b{}^a}{\bar f}\frac{\partial \bar f}{\partial y^d} + \frac{\eta_{bd}}{\bar f}\eta^{as}\frac{\partial \bar f}{\partial y^s} \tag{3.89}$$

$$\Gamma^a_{ad} = -\frac{4}{\bar f}\frac{\partial \bar f}{\partial y^d} \tag{3.90}$$

and $T_{(4)b\ldots;t}^{a\ldots}$, the covariant derivative of $T_{(4)b\ldots}^{a\ldots}$, is given by

$$T_{(4)b\ldots;t}^{a\ldots} = T_{(4)b\ldots;t}^{a\ldots} - W\Gamma^d_{dt}T_{(4)b\ldots}^{a\ldots} + \Gamma^a_{dt}T_{(4)b\ldots}^{d\ldots} +$$
$$+ \ldots - \Gamma^d_{bt}T_{(4)d\ldots}^{a\ldots} - \ldots \tag{3.91}$$

Thus

$$T^{a\ldots}_{b\ldots t}\Big|_{\substack{\Phi=0\\ \omega=1}}$$

fails to be the covariant derivative of $T_{(4)b\ldots}^{a\ldots}$ due to the presence of the term

$$-\frac{m}{\bar f}\frac{\partial \bar f}{\partial y^a}T_{(4)b\ldots}^{a\ldots}$$

and due to the fact that $-K_d{}^A K^a_{A,T} K_t{}^T$ contains only the first two terms of Γ^a_{dt}. This is unlike what occurs in many other previously studied cases of projections. Geroch has studied the projection into quotient spaces with respect to Killing vector fields (Geroch, 1971 and 1972) and Bergmann has studied the same question with respect to unit time like vector fields (Bergmann, 1942) and in both cases their equations analogous to (3.83) give the covariant derivative in the quotient space. Israel has studied the projection into three-dimensional space like hypersurfaces and again the equation analogous to (3.83) gives the covariant derivative generated by the projection of the metric (Israel, 1966).

If we fix $m = W$ in (3.82) and define a new affinity on R_6 given in the x^A coordinate system by

$$\Lambda^A_{BD} = \eta_{BD}\eta^{AF}K_F{}^G(\log\Omega)_{,G} \tag{3.92}$$

and define $T^{A\ldots}_{B\ldots|T}$ as the covariant derivative calculated using Λ^A_{BD} then

$$T_{(4)b\ldots;t}^{a\ldots} \underset{\substack{\Phi=0\\ \omega=1}}{=} J_0{}^W \bar f^W K_A{}^a K_b{}^B \ldots K_t{}^T T^{A\ldots}_{B\ldots|T} \tag{3.93}$$

It does not seem clear why this value of m is required and why R_6 has to carry a non-metric generated affinity to get the C_4 covariant derivative in terms of a R_6 derivative.

For the case of projecting into M_4, Λ^A_{BD} is easily calculated to be zero. It also follows directly that if we define G_{AB}, the K type tensor that represents the C_4 metric,

$$G_{AB} \equiv K_A{}^E K_B{}^F \eta_{EF} \tag{3.94}$$

then

$$K_A{}^E K_B{}^F K_T{}^H G_{EF|H} = 0 \tag{3.95}$$

as would be expected.

4. *Four-Space Field Equations Imbedded in Six-Space*

With the formalism that has been developed in the preceding section it is straightforward to imbed a four-space equation in six-space. It will be sufficient for our purposes to just consider the mapping to be from M_4 to R_6. In this case

$$\omega \equiv f(x^a, \Omega) = \Omega \tag{4.1}$$

$$\frac{1}{2f}\frac{\partial f}{\partial x^A} = \tau_A \tag{4.2}$$

$$K_a{}^A = \Omega(\delta_a{}^A - \tau^A \Phi_a) \tag{4.3}$$

$$K_A{}^a = \frac{1}{\Omega}(\delta_A{}^a - \Phi^a \tau_A) \tag{4.4}$$

Some other useful formulas for this section are:

$$\tau_{A,B} = -2\tau_A \tau_B \tag{4.5}$$

$$K^a_{A,B} = -4\tau_{(A} K^b_{B)} \tag{4.6}$$

$$K^A_{a,B} = -2\tau_B K_a{}^A - 2\tau^A \eta_{BT} K_a{}^T \tag{4.7}$$

All equations in this section will use these specializations to flat space.

We first look at the imbedding of the zero mass Klein-Gordon equation, which was first studied by Dirac (Dirac, 1936). Starting with the M_4 equation

$$\eta^{ab}\frac{\partial}{\partial y^a}\left(\frac{\partial}{\partial y^b} S_{(4)}\right) = 0 \tag{4.8}$$

$S_{(4)}$ being a scalar density of weight W we take S as our R_6 representation of $S_{(4)}$, where

$$S = S_0 + \Phi S_1 + \dots$$
$$S_0 = \omega^m S_{(4)} \tag{4.9}$$

That is

$$S_0(x^A) = J_0^{-W}\Omega^m S_{(4)}\left(\frac{x^a}{\Omega}\right)$$
$$= 4^W \Omega^{-3W+m} S_{(4)}(x^a/\Omega) \tag{4.10}$$

and S is homogeneous of degree $\nu = -3W + m$. Since

$$\eta^{ab} = \Omega^2 K_A{}^a K_B{}^b \eta^{AB} \tag{4.11}$$

$$\frac{\partial}{\partial y^a} = K_a{}^A \frac{\partial}{\partial x^A} \tag{4.12}$$

$$\Omega_{,A} K_a{}^A = \Phi_{,A} K_a{}^A = 0 \tag{4.13}$$

equation (4.8) reads on N_5

$$\Omega^2 \eta^{MN} K_M{}^a K_N{}^b K_a{}^A \{K_b{}^B [\Omega^{-\nu}(S_0 + \Phi S_1 + \ldots)]_{,B}\}_{,A} = 0 \tag{4.14}$$

Expanding equation (4.14) gives

$$\eta^{AB} S_{,AB} - 4(\nu + 1)\tau^A S_{,A} = 0 \tag{4.15}$$

on N_5.

We notice that for any S_n, homogeneous of degree $\nu - 2n$ and just a function of (x_a, Ω), $\Phi^n S_n$ satisfies equation (4.15) on N_5. For $\nu = -1$ (equation 4.8) is thus equivalent to the equation set

$$\begin{aligned} \eta^{AB} S_{,AB} &= \Phi T \\ S_{,A} \Phi^A &= -2S \end{aligned} \tag{4.16}$$

where T is arbitrary. This set of equations is clearly invariant under $SO(4, 2)$. It will be necessary to carefully discuss later how this is related to the invariance under the C_4 conformal group of equation (4.8) for $W = \frac{1}{4}$. It is important to note that as we are interested in using the six-dimensional formalism to study conformal invariance it would not have been useful to imbed by taking all terms but S_0 equal zero in equation (4.9) as this would have added a further constraint condition to the set (4.16), namely

$$S_{,A} \tau^A = 0 \tag{4.17}$$

and this condition is not satisfied by $\mathscr{L}_{\xi A} S$ even modulo equation set (4.16). In general when imbedding M_4 equations in R_6 only constraint equations that do not break invariance under $SO(4, 2)$ should be chosen.

The mass Kline-Gordon equation, with $\nu = -1$, takes the form on N_5

$$\begin{aligned} &\Omega^2 \eta^{MN} K_M{}^a K_N{}^b K_a{}^A \{K_b{}^B [\Omega^{-\nu}(S_0 + \Phi S_1 + \ldots)]_{,B}\}_{,A} \\ &\qquad + m^2 \Omega^{-\nu}(S_0 + \Phi S_1 + \ldots) = 0 \end{aligned} \tag{4.18}$$

which again expanding out gives

$$\begin{aligned} \eta^{AB} S_{,AB} + \frac{m^2}{\Omega^2} S &= \Phi T \\ S_{,A} \Phi^A &= -2S \end{aligned} \tag{4.19}$$

so that invariance under $SO(4, 2)$ is broken by the Ω^{-2} multiplying the m^2 term in equation (4.19). Equation set (4.19) is of course invariant under the subgroup of $SO(4, 2)$ that is mapped into the Poincaré group.

As another example we imbed the dilation invariant, but non-conformally invariant M_4 equation

$$\eta^{ab} \frac{\partial}{\partial y^a}\left(\frac{\partial S_{(4)}}{\partial y^b}\right) + \eta^{ab} \frac{\partial S}{\partial y^a} \frac{\partial S}{\partial y^b} = 0 \tag{4.20}$$

in R_6. We notice that for this equation to be well defined $S_{(4)}$ must be a scalar density of weight 0, and S must be taken to be homogeneous of degree 0. Equation (4.20) then goes to

$$\begin{aligned} \eta^{AB}(S_{,AB} + S_{,A}S_{,B}) + 4\tau^A(SS_{,A} - S_A) = \Phi T \\ S_{,A}\Phi^A = 0 \end{aligned} \tag{4.21}$$

Equation set (4.21) is non-invariant under $SO(4, 2)$ due to the non-removable presence of a term containing τ_B. In general as

$$\tau^A_{,B} = -2\tau^A \tau_B, \qquad \Omega_{,A} = +2\Omega\tau_A \tag{4.22}$$

M_4 equations imbedded in R_6 will be non-invariant under $SO(4, 2)$ due to the presence only of non-removable factors of Ω, or τ_A. The factors of Ω arise from mass type terms, and the τ^A factors come from derivative couplings. It is clear they are of a fundamentally different nature. The presence of τ^A in the R_6 equation is related to the non-variance of the M_4 equation under the moving of infinity in M_4, while the presence of Ω is related to non-invariance under scaling. It should be noted that the Lie derivative of τ^A with respect to the ξ^A that represents the dilations equation (3.14) is zero, so that imbedded equations that have only τ^A in them will be dilation invariant. We thus have that any type of broken conformal invariance will correspond to the presence of only two distinct types of terms in the R_6 equations.

As an illustration of the imbedding of tensor fields we will imbed Maxwell's equations (equations 2.21 and 2.22 in R_6. This again was first done by Dirac (1936). We represent $F_{(4)ab}$ by a homogeneous, of degree f, anti-symmetric six-space tensor F_{AB}. F_{AB} can be written as

$$F_{AB} = F_{ab}K_A{}^aK_B{}^b + E_aK^a_{[A}\tau_{B]} + H_aK^a_{[A}\Phi_{B]} + I\tau_{[A}\Phi_{B]} \tag{4.23}$$

where

$$F_{AB}K_a{}^A K_b{}^B = F_{0ab} + \Phi F_{1ab} + \ldots \tag{4.24}$$

$$F_{(4)ab} = \Omega^{-p} F_{ab|\Phi=0} \tag{4.25}$$

$$p = f + 2 \tag{4.26}$$

and E_a, H_a and I are not determined by $F_{(4)ab}$. It is clear that we can impose everywhere the $SO(4, 2)$ invariant condition

$$F_{AB}\Phi^B = 0 \tag{4.27}$$

which implies defining

$$B_A \equiv H_aK_A{}^a = F_{AB}\tau^B \tag{4.28}$$

that

$$F_{AB} = F_{ab}K_A{}^a K_B{}^b - 4\Phi B_{[A}\tau_{B]} + B_{[A}\Phi_{B]} \tag{4.29}$$

In the same way as for the Klein–Gordon equation, we get the R_6 equation set (4.30) equivalent to (2.21) to be

$$\begin{aligned} \eta^{BD}F_{AB,D} &= -2[f+2]F_{AB}\tau^B - F_{MB,D}\eta^{BD}\Phi_A\tau^M \\ &= \Phi T_A \\ F_{AB,C}\Phi^C &= 2fF_{AB} \\ F_{AB}\Phi^B &= 0 \end{aligned} \tag{4.30}$$

For $f = -2$, we have the first equation of the set as

$$\eta^{BD}F_{AB,D}[\delta_T{}^A - \tau^A\Phi_T] = \Phi T_A \tag{4.31}$$

We of course desire the stronger condition

$$\eta^{BD}F_{AB,D} = \Phi T'_A \tag{4.32}$$

as our representation of the M_4 Maxwell equation. For equation (4.32) to hold it is necessary that not only $F_{(4)ab}$ satisfy equation (2.21), but also that

$$B^A{}_{,A} = \Phi D \tag{4.33}$$

Since $\mathscr{L}_{\xi E}\, F_{AB}$ satisfies equation (4.27) it can be again expanded in the form of equation (4.29), and then since it also satisfies equation (4.32) it will have its corresponding element B'_A satisfying

$$B'^A{}_{,A} = \Phi D \tag{4.34}$$

so that condition (4.33) does not break $SO(4, 2)$ invariance. Thus we have that equation (2.21) on M_4 is equivalent to the $SO(4, 2)$ invariant set of equations

$$\begin{aligned} \eta^{BC}F_{AB,C} &= \Phi T'_A \\ F_{AB,C}\Phi^C &= -4F_{AB} \\ F_{AB}\Phi^B &= 0 \end{aligned} \tag{4.35}$$

With the constraint conditions of equation set (4.35) along with the further imbedding condition

$$B_{D,A} - B_{A,D} - 4B_A\tau_D + 4B_D\tau_A + F_{lab}K^a_{[A}K^b_{B]} = \Phi W_{AB} \tag{4.36}$$

Equation (2.22) on M_4 is equivalent to

$$F_{[AB,C]} = \Phi T_{ABC} \tag{4.37}$$

Condition (4.36) is again allowable due to equation (4.37).

It is interesting to note that the homogeneity condition which puts Maxwell's M_4 equations in their $SO(4, 2)$ invariant form on R_6 is the one that puts on N_5

$$\mathscr{L}_{\nu A}\, F_{AB} = 0 \tag{4.38}$$

This is unlike the case of the zero-mass Klein-Gordon equation with $W = +\frac{1}{4}$. In many cases of imbedding in projective spaces one assumes a relationship such as equation (4.38), as the basic condition of imbedding.

As we have said before it is necessary to carefully examine the relation between six-dimensional equations that are invariant under $SO(4, 2)$, and the invariance of the corresponding four-dimensional ones under the conformal group. Invariance is taken again in the sense of equation (2.7). That care is needed for this examination is shown by the fact that a M_4 scalar field of any weight, satisfying equation (4.8) can be mapped into equation set (4.16), but only for $W = \frac{1}{4}$ is the M_4 equation conformally invariant. Also with the non-invariant $SO(4, 2)$ constraint equation (4.17), equation (4.20) clearly can be represented as a conformally invariant equation.

To study this question we first need for tensor densities on M_4 and R_6 respectively the relationship between the four-dimensional Lie derivative with respect to ξ^a, and the six-dimensional Lie derivative with respect to ξ^A. For a tensor density of weight W on M_4

$$\underset{\xi}{\mathscr{L}_d}\, T_{(4)b\ldots}^{\ a\ldots} = \frac{\partial T_{(4)b\ldots}^{\ a\ldots}}{\partial y^t}\,\xi^t - T_{(4)b\ldots}^{\ t\ldots}\,\xi^a_{,t} - \ldots + T_{(4)t\ldots}^{\ a\ldots}\,\xi^t_{,b} + \ldots + W T_{(4)b\ldots}^{\ a\ldots}\,\frac{\partial \xi^t}{\partial y^t} \tag{4.39}$$

A tensor density $T_{B\ldots}^{A\ldots}$ of weight W and homogeneous of degree t on R_6 can be expanded in a basis set consisting of $K_a{}^A$, τ^A, μ^A and its dual basis, $K_A{}^a$, Φ_A, τ_A, where

$$\mu^A = 2\Omega\nu^A \tag{4.40}$$

Then

$$T_{B\ldots}^{A\ldots} = \tilde{T}_{b\ldots}^{a\ldots}\,K_a{}^A K_B{}^b \ldots + T_{b\ldots}^{\tau\ldots}\,\tau^A K_B^b \ldots + \ldots + T_{\tau\ldots}^{a\ldots}\tau_B K_a^A \ldots + \ldots + T_{b\ldots}^{\mu\ldots}\,K_B{}^b \mu^A \ldots + \ldots + T_{\Phi\ldots}^{a\ldots}\,K_a{}^A \Phi_B \ldots + \ldots \tag{4.41}$$

$$\overset{K}{T}{}_{B\ldots}^{A\ldots} = \tilde{T}_{b\ldots}^{a\ldots}\,K_a{}^A K_B{}^b \ldots \tag{4.42}$$

and

$$4^{-W}\Omega^{-(t-k+l)}\tilde{T}_{0b\ldots}^{\ a\ldots} = T_{(4)b\ldots}^{\ a\ldots} \tag{4.43}$$

$$T_{B\ldots}^{A\ldots}\Phi_A = T_{B\ldots}^{A\ldots}\Phi^B = \ldots = 0 \tag{4.44}$$

if and only if in expansion (4.41) all coefficients multiplying terms with τ^A in them are zero.

$$\underset{\xi^E}{\mathscr{L}}\, T_{B\ldots}^{A\ldots} = T_{B\ldots,R}^{A\ldots}\,\xi^R - T_{B\ldots}^{R\ldots}\,\xi^A_{,R} + T_{R\ldots}^{A\ldots}\,\xi^R_{,B} + \ldots \tag{4.45}$$

As

$$\xi^A_{,A} = 0 \tag{4.46}$$

it does not appear in equation (4.45). To relate equations (4.39) and (4.45) we also note that

$$\underset{\xi^A}{\mathscr{L}} \Phi = 0 \tag{4.47}$$

$$\underset{\xi^R}{\mathscr{L}} \mu^A = -4\Phi \underset{\xi^R}{\mathscr{L}} \tau^A \tag{4.48}$$

$$\underset{\xi^R}{\mathscr{L}} \tau^A = -2\tau^A \tau_B \xi^B + \xi^A_{,B}\tau^B \tag{4.49}$$

It is direct to calculate that

$$\xi^A_{,B}\tau^B K_A{}^a|_{\Phi=0} \neq 0 \tag{4.50}$$

We then can use equations (3.57) and (3.59) which relate ξ^a and ξ^A and give

$$\frac{\partial \xi^a}{\partial y^a} = -8\tau_B \xi^B \tag{4.51}$$

to imbed equation (4.39) in R_6. The result is

$$\begin{aligned}\underset{\xi^R}{\mathscr{L}} \overset{K}{T}{}^{A\ldots}_{B\ldots} = 4^{+W}\Omega^{(t-k+l)}\{(\underset{\xi^a}{\mathscr{L}} T^{a\ldots}_{b\ldots})K_a{}^A K_B{}^b \ldots \\ + 2(4W+t-k+l)\tau_B \xi^B T^{a\ldots}_{b\ldots} K_a{}^A K_B{}^b \ldots + S^{A\ldots}_{B\ldots}\}\end{aligned} \tag{4.52}$$

$$S^{A\ldots}_{B\ldots} K_A{}^a K_b{}^B \ldots|_{\Phi=0} = 0 \tag{4.53}$$

We thus have that for

$$W = -\frac{(t+l-k)}{4} \tag{4.54}$$

and equation (4.44) holding

$$\left((\underset{\xi^R}{\mathscr{L}} T^{A\ldots}_{B\ldots})K_A{}^a K_b{}^B \ldots\right)_{\Phi=0} = 4^W \Omega^{(t-k+l)} \underset{\xi^r}{\mathscr{L}} T^{a\ldots}_{b\ldots} \tag{4.55}$$

If $T^{A\ldots}_{B\ldots}$ satisfies a $SO(4, 2)$ invariant set of equations, that is it contains no absolute objects whose Lie derivative with respect to ξ^A is non zero,

$$T'^{A\ldots}_{B\ldots} = T^{A\ldots}_{B\ldots} + \underset{\xi^R}{\mathscr{L}} T^{A\ldots}_{B\ldots}$$

is also a solution to the equations. $T'^{A\ldots}_{B\ldots}$ can be expanded in the same way as $T^{A\ldots}_{B\ldots}$ to give

$$T'^{A\ldots}_{B\ldots} = \tilde{T}'^{a\ldots}_{b\ldots} K_a{}^A K_B{}^b \ldots + \ldots \tag{4.56}$$

If this set of equations is equivalent to a M_4 set of equations then $\overset{K}{T}{}'^{A\ldots}_{B\ldots}|_{\Phi=0}$ will be a solution to the equations. If equation (4.55) holds also then

$\mathscr{L}_\xi T^{a\cdots}_{b\cdots} + T^{a\cdots}_{b\cdots}$ will be a solution to the corresponding M_4 equations.

We thus have the theorem. If the M_4 field equations satisfied by $T_{(4)}{}^{a\cdots}_{b\cdots}$, a tensor density of weight $\bar{W}$, can be imbedded as a $SO(4, 2)$ invariant set of field equations in R_6, satisfied by $T^{A\cdots}_{B\cdots}$, which is homogeneous of degree t, and in which equation (4.43) is one of the constraint equations, then the M_4 field equations are invariant under the conformal group for a tensor density of weight $W = -(t + l - k)/4$.

Equation (4.54) clearly gives the relationship between the weight that makes the M_4 equation conformally invariant and the degree of homogeneity that makes the imbedded equation $SO(4, 2)$ invariant. From (3.80) it follows that the condition to make the imbedded equation $SO(4, 2)$ invariant and the condition which makes the Lie derivative with respect to v^A vanish on N_5 agree for any 0 weight tensor and for no other kind.

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References

Anderson, J. L. (1967). *Relativity Physics.* Academic Press, New York.
Barut, A. O. and Haugen, R. B. (1972). *Annals of Physics,* 71, 519.
Bergmann, P. G. (1942). *Introduction to the Theory of Relativity,* Chapter XVII, Prentice-Hall Inc., Englewood Cliffs, N.J.
Carruthers, P. (1971). *Physics Reports,* 1, 1.
Dirac, P. A. M. (1936). *Annals of Mathematics,* 37, 429.
Eisenhart, L. P. (1927). *Non Riemannian Geometry*, Chapter 4. American Mathematical Society, New York.
Geroch, R. (1971). *Journal of Mathematical Physics,* 12, 918; (1972). *Journal of Mathematical Physics,* 13, 394.
Grgin, E. (1968). *Journal of Mathematical Physics,* 9, 1595.
Israel, W. (1966). *Nuovo Cimento B,* 44, 1.
Kastrup, H. A. (1966). *Physical Review,* 150, 1183.
Mack, G. and Salam, A. (1969). *Annals of Physics,* 53, 174.
Penrose, R. (1963). In *Relativity, Groups and Topology,* p. 565. Gordon and Breach, New York.
Yano, K. (1965). *The Theory of Lie Derivatives.* North-Holland, Amsterdam.